Average Resonance Parameters and the Optical Model*

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Averages over resonances of the elements of the collision matrix are derived from R-matrix theory for the general many-channel case. Of special interest are the average diagonal elements because of their relation to the complex phase shifts of optical models for the channels. In the absence of correlations among channel width amplitudes, the formula relating an average diagonal element to average resonance parameters is found to be the same as obtained by Thomas for the single-channel case, or when competing channels have small transmission factors. The evaluation of this formula with the aid of the single-particle level picture of Lane, Thomas, and Wigner is discussed, as well as the effect of correlations among channels which are indicative of direct reaction effects.

INTRODUCTION

IT has previously been pointed out[1] that the comparison of average total and scattering cross-section data, on the one hand, with average compound nucleus reaction cross sections or resonance data, on the other, requires a knowledge of the relationship between optical-model parameters (such as transmission coefficients) and average channel width to resonance spacing ratios. This relationship was derived in reference 1 on the basis of an unjustified assumption, according to which the contributions from all resonances outside the averaging interval were thrown into an unevaluated and neglected contribution. However, this contribution is, in fact, not always negligible and, further, bears a definite relation to the optical-model parameters. The purpose of the following is to obtain an improved relationship. The resulting expression is, in essence, identical to one obtained by Thomas[2] employing the channel elimination method and assuming that the width to spacing ratios of the eliminated channels are small. The more general validity of the relation which is provided here is important, first, because it demonstrates the validity of the optical-model assumption regarding the independence of the average formation cross section from the decay processes and, secondly, because the assumption that all other channels have small width to spacing ratios is violated in many important applications.

The formula obtained is explored in the light of the interpretation of the optical model given by Lane, Thomas, and Wigner,[3] and the effect of direct interactions is demonstrated.

The notation employed follows that in the review paper by Lane and Thomas,[4] to which the reader is referred for definitions of all unexplained notation.

THE AVERAGE COLLISION MATRIX

In the R matrix formalism the collision matrix U is given by the expressions[5]

$$\mathbf{U}=\boldsymbol{\Omega}\boldsymbol{\mathfrak{P}}^{1/2}(\mathbf{1}-\mathbf{RL}^0)^{-1}(\mathbf{1}-\mathbf{R}\boldsymbol{\mathfrak{L}}^0)\boldsymbol{\mathfrak{P}}^{-1/2}\boldsymbol{\Omega} \tag{1a}$$

$$=\boldsymbol{\Omega}\{\mathbf{1}-(\mathbf{L}^0)^{-1}\boldsymbol{\mathfrak{P}}\mathbf{w}+\boldsymbol{\mathfrak{P}}^{1/2}(\mathbf{L}^0)^{-1}\times[(\mathbf{L}^0)^{-1}-\mathbf{R}]^{-1}(\mathbf{L}^0)^{-1}\boldsymbol{\mathfrak{P}}^{1/2}\mathbf{w}\}\boldsymbol{\Omega}. \tag{1b}$$

If the consideration of $\mathbf{U}$ and the resulting cross sections can be limited to an interval ΔE of the total energy of the system such that

$$\Delta E \ll S_c(dS_c/dE)^{-1} \quad \text{for all channels } c, \tag{2}$$

then Eqs. (1) can be simplified by choosing the boundary conditions B_c for all channels to be equal to the shift functions S_c so that $L_c^0=iP_c$.[6] Then Eq. (1b) becomes

$$\mathbf{U}=\boldsymbol{\Omega}\{2[\mathbf{1}-i\mathbf{P}^{1/2}\mathbf{R}\mathbf{P}^{1/2}]^{-1}-\mathbf{1}\}\boldsymbol{\Omega}. \tag{3}$$

We now wish to average the elements of $\mathbf{U}$ over an energy interval ΔE which first of all satisfies the inequality (2), and secondly contains many resonances (poles of $\mathbf{R}$)

$$\Delta E \gg D, \tag{4}$$

where D is the average spacing of resonances. We shall also assume that the averages of $\gamma_{\lambda c}$ and the level spacing are almost constant in ΔE. The average of $\mathbf{U}$ is most easily performed by a method of Thomas.[2] One observes that the elements of $\mathbf{U}$ are meromorphic in the complex energy plane and that (with the exception of the ground-state pole on the real axis) all poles are in the lower half plane.[4] Then the averages

$$\langle U_{cc'}\rangle_{\mathrm{av}}=\frac{1}{\Delta E}\int_{\Delta E} dE\, U_{cc'} \tag{5}$$

are performed by considering the vanishing integral of $U_{cc'}$ over the contour shown in Fig. 1. If $\mathcal{E}\gg D$, it is easily seen that $\mathbf{R}$ is essentially constant along the

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[1] P. A. Moldauer, Phys. Rev., **123**, 968 (1961).
[2] R. G. Thomas, Phys. Rev., **97**, 224 (1955).
[3] A. M. Lane, R. G. Thomas, and E. P. Wigner, Phys. Rev. **98**, 693 (1955).
[4] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958).
[5] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947). Expression (1b) corrects typographic errors in Eq. (VII, 1.6b) of reference 4.
[6] For tabulations of S_c, P_c, and Ω_c for neutron channels see J. E. Monahan, L. C. Biedenharn, and J. P. Schiffer, Argonne National Laboratory Report ANL-5846 (unpublished).

upper horizontal piece of the contour and the fluctuations in $\mathbf{R}(E+i\mathcal{E})$ can be made arbitrarily small by making $\mathcal{E}/D$ sufficiently large. We shall also assume that ΔE and $\mathcal{E}$ can be chosen so that along any horizontal line within the contour the variation in L_c^0 satisfies $\Delta L_c^0/L_c^0 \ll 1$, for all c. These conditions are consistent with condition (4) for nucleon-induced reactions in almost all medium and heavy nuclei, except very close to thresholds, particularly if the threshold channels include s-wave neutron emission. At energies sufficiently close to an s-wave neutron threshold that these conditions cannot be met, the variation of L_c is so severe that averages will depend on the precise energy interval ΔE which is chosen and, correspondingly, average cross-section measurements will depend on the chape of the incident flux. The interval must *not*, of course, *include* a threshold.[7]

With these assumptions the contributions to the contour integral form the two vertical paths cancel except for a small remainder from the region of strong fluctuations near the real axis. The resulting ΔE dependent contribution to $\langle \mathbf{U} \rangle_{\text{av}}$, can be eliminated by defining the $\langle U_{cc'} \rangle_{\text{av}}$ as the average over many expressions (5) whose intervals ΔE are slightly displaced. Then, since there are no singularities within the contour, $\langle U_{cc'} \rangle_{\text{av}}$ is equal to the average along the upper horizontal path. Since $\mathbf{R}$ is constant there, so is $\mathbf{U}$ and hence[8]

$$\langle U_{cc'} \rangle_{\text{av}} = U_{cc'}(E+i\mathcal{E}). \tag{6}$$

With Eq. (2) satisfied along the upper horizontal path we may calculate $\langle U_{cc'} \rangle_{\text{av}}$ by evaluating Eq. (3) at $E+i\mathcal{E}$. This requires the evaluation of

$$\mathbf{R}(E+i\mathcal{E}) = \sum_\lambda \frac{\gamma_\lambda \times \gamma_\lambda}{E_\lambda - E - i\mathcal{E}} = \int_{-\infty}^{\infty} dE'\, \rho(E') \frac{\langle \gamma \times \gamma \rangle_{E'}}{E' - E - i\mathcal{E}}, \tag{7}$$

where $\rho(E)$ is the resonance level density at E and the brackets $\langle\ \rangle_E$ indicate an average over resonances in the vicinity of E. For the diagonal elements of $\mathbf{R}$ we obtain from Eq. (7)

$$R_{cc}(E+i\mathcal{E}) = i\pi \frac{\langle \gamma_c^2 \rangle_E}{D(E)} + \tilde{R}_c, \tag{8}$$

where

$$\tilde{R}_c = \int_{-\infty}^{\infty} dE'\, \rho(E') \langle \gamma_c^2 \rangle_E \frac{E'-E}{(E'-E)^2 + \mathcal{E}^2}. \tag{9}$$

In order to evaluate expressions (8) and (9) we use the optical model for the strength function

$$s_c(E) = \rho(E) \langle \gamma_c^2 \rangle_E. \tag{10}$$

Particularly convenient for this purpose is the formulation of Lane, Thomas, and Wigner,[3] according to which s_c consists of a sum of contributions from single-particle levels at energies E_p with strengths ζ_{cp}^2 and widths W_p so that

$$s_c(E) = (1/2\pi) \sum_p \zeta_{cp}^2 W_p [(E_p - E)^2 + \tfrac{1}{4} W_p^2]^{-1}. \tag{11}$$

Substituting Eq. (11) into Eq. (7) one obtains, assuming $\mathcal{E} \ll W_p$,

$$R_{cc}(E+i\mathcal{E}) = \sum_p \zeta_{cp}^2 [(E_p - E) - iW_p]^{-1} = i\pi s_c(E) + \sum_p \zeta_{cp}^2 \frac{(E_p - E)}{(E_p - E)^2 + \tfrac{1}{4} W_p^2}. \tag{12}$$

The second term can conveniently be split into two parts, one arising from one or two nearby single-particle or optical-model resonances n and the other from far away resonances f. Then

$$\tilde{R}_c = \tilde{R}_c^0 + \tilde{R}_c^\infty, \tag{13}$$

where

$$\tilde{R}_c^\infty \sim \sum_f \frac{\zeta_{cf}^2}{E_f - E} \sim \int_{E_0}^{\infty} \frac{dE'\, \rho_{\text{sp}}(E') \zeta_c^2(E')}{E' - E}. \tag{14}$$

Here E_0 is the energy of the lowest single-particle level, ρ_{sp} is the density of single-particle levels and it has been assumed that $\rho_{\text{sp}} \zeta_c^2$ is constant in the vicinity of E. It is clear that this quantity cannot remain constant if $\tilde{R}_c^\infty$ is to be finite.

The contribution of a nearby optical-model resonance n to $\tilde{R}_c^0$ is

$$\tilde{R}_{cn}^0(E) = 2\pi s_{cn}(E)(E_n - E)/W_n, \tag{15}$$

where s_{cn} is the contribution of resonance n to the strength function. Expression (15) is a maximum when $|E - E_n| = \frac{1}{2} W_n$ and then amounts to $\pi s_{cn}(E)$.

In order to evaluate the average collision matrix, we shall at first assume that

$$\langle \gamma_{\lambda c} \gamma_{\lambda c'} \rangle_E = 0 \quad \text{for} \quad c' \neq c, \tag{16}$$

and for all energies E. This assumption will be further discussed in the next section where it is shown that (16) implies the presence of pure compound nucleus reactions only. According to Eq. (7), the assumption (16) means

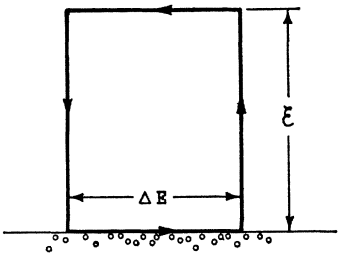

FIG. 1. Path of integration for the averaging of the elements of the collision matrix by the method of Thomas. The circles indicate poles of the collision matrix.

[7] For applications of these methods to average cross sections in the vicinity of thresholds see W. E. Meyerhof (to be published).

[8] The fluctuations in $U(E+i\mathcal{E})$ produced by small residual fluctuations in $R(E+i\mathcal{E})$ are easily estimated by means of perturbation methods and are found to be small to the same order.

that $\mathbf{R}(E+i\mathcal{E})$ is diagonal and, hence, Eqs. (3) and (6) are easily evaluated with the aid of definitions (8) and (10) to yield

$$\langle U_{cc}(E)\rangle_{\rm av}=0 \quad \text{for} \quad c'\neq c, \tag{17a}$$

$$\langle U_{cc}(E)\rangle_{\rm av}=\Omega_c^2\frac{1-\pi P_c s_c+iP_c\tilde{R}_c}{1+\pi P_c s_c-iP_c\tilde{R}_c}=e^{+2i\delta_c}, \tag{17b}$$

where δ_c is the complex optical-model scattering phase shift.[9] Introducing the resonance parameter[1]

$$\langle\tau_c\rangle=2\pi\langle\Gamma_c\rangle/D=4\pi P_c s_c, \tag{18}$$

one obtains for the transmission coefficient

$$T_c=1-|\langle U_{cc}\rangle_{\rm av}|^2=\langle\tau_c\rangle[(1+\tfrac{1}{4}\langle\tau_c\rangle)^2+(P\tilde{R}_c)^2]^{-1}. \tag{19}$$

If we assume that $\tilde{R}_c$ arises only from the nearest optical-model resonance, then according to Eqs. (15) and (18) $P\tilde{R}_c$ reaches the maximum value of $\frac{1}{4}\langle\tau_c\rangle$ at one half-width from the optical-model peak and decreases to zero at the peak. One interesting consequence of this assumption is that T_c can reach its theoretical maximum value of unity at the optical-model peak (if $\langle\tau_c\rangle=4$ there), but cannot be greater than $2(1+\sqrt{2})^{-1}=0.828$ at the half maximum (if $\langle\tau_c\rangle=2\sqrt{2}$ there).

COMPOUND NUCLEUS AND DIRECT REACTIONS

The assumption (16) is clearly satisfied if at all energies the $\gamma_{\lambda c}$ are randomly distributed with zero mean in λ and if the random sequences $\gamma_{\lambda c}$ and $\gamma_{\lambda c'}$ are uncorrelated. These are the customary statistical assumptions regarding the amplitudes of compound nucleus resonances,[4] implying the statistical independence of resonance states and the lack of phase correlations among different decay modes. Furthermore, unless the assumption (16) is violated for some pair of channels c and c', the matrix element $R_{cc'}=\sum_\lambda(\gamma_{\lambda c}\gamma_{\lambda c'})(E_\lambda-E)^{-1}$ cannot contain a part which varies smoothly with energy, since such a part could arise only from correlated contributions of distant resonances. It follows from Eq. (3) that also $U_{cc'}$ can only exhibit long-range correlations in energy—and hence direct reaction effects—if Eq. (16) and hence also Eq. (17a) is violated.

[9] H. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1954).

To see the effect of direct reactions upon the results of the preceding section, we assume that Eq. (16) is violated only for the pair of channels c and c' and we let $R_{cc'}(E+i\mathcal{E})=\mathcal{R}_{cc'}(E)=\mathcal{R}_{c'c}(E)$ which is assumed to be calculable by a direct reaction model, while all other off-diagonal elements of $\tilde{R}(E+i\mathcal{E})$ still vanish. Then evaluation of $\langle\mathbf{U}\rangle_{\rm av}$ requires the inversion of a 2×2 matrix for those elements involving only channels c and c' while all other elements remain unaffected. In particular we obtain

$$\langle U_{cc}\rangle_{\rm av}=\Omega_c^2\left[1-\tfrac{1}{4}\langle\tau_c\rangle+iP_c\tilde{R}_c-\frac{P_{c'}\mathcal{R}_{cc'}{}^2P_c}{1+\tfrac{1}{4}\langle\tau_{c'}\rangle-iP_{c'}\tilde{R}_{c'}}\right]$$
$$\times\left[1+\tfrac{1}{4}\langle\tau_c\rangle-iP_c\tilde{R}_c+\frac{P_{c'}\mathcal{R}_{cc'}{}^2P_c}{1+\tfrac{1}{4}\langle\tau_{c'}\rangle-iP_{c'}\tilde{R}_{c'}}\right]^{-1}, \tag{20}$$

and

$$\langle U_{cc'}\rangle_{\rm av}=2i\Omega_c\Omega_{c'}(P_cP_{c'})^{1/2}\mathcal{R}_{cc'}$$
$$\times[(1+\tfrac{1}{4}\langle\tau_c\rangle-iP_c\tilde{R}_c)(1+\tfrac{1}{4}\langle\tau_{c'}\rangle-iP_{c'}\tilde{R}_{c'})+P_{c'}\mathcal{R}_{cc'}{}^2P_c]^{-1}. \tag{21}$$

We note that in the absence of direct reactions Eq. (17) shows that $\langle U_{cc}\rangle_{\rm av}$ is independent of parameters describing events in channels c' and that the same is true of the connection between optical-model and resonance parameters as shown by Eq. (19). As a consequence also, the optical-model parameters may be expected to be uninfluenced by channels c'. This assures, furthermore, that on the average, compound nucleus formation is independent of the possible modes of decay. It does *not* mean, however, that decay in any *particular* channel is on the average independent of formation (cf. reference 1).

In the presence of direct reactions, as shown by Eq. (20), the relation between optical-model parameters and resonance parameters in channel c, involves not only the parameters of the direct process $\mathcal{R}_{cc'}$ but also the resonance parameters $\langle\tau_{c'}\rangle$, $P_{c'}\tilde{R}_{c'}$ of the channel c'.

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